

PATTERN RECOGNITION ON ORIENTED MATROIDS: SYMMETRIC CYCLES IN THE HYPERCUBE GRAPHS

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ABSTRACT. If \mathfrak{V} is the vertex sequence of a symmetric $2t$ -cycle in the hypercube graph with the vertices $\{1, -1\}^t$, then for any vertex T of the graph there exists a unique inclusion-minimal subset of \mathfrak{V} such that T is the sum of its elements. We present a simple combinatorial statistic on decompositions of vertices of the hypercube graphs with respect to symmetric cycles and describe their basic metric properties.

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1. INTRODUCTION

Let $\mathcal{H} := (E_t, \{1, -1\}^t)$ be the oriented matroid, on the ground set $E_t := \{1, \dots, t\}$, whose set of topes $\{1, -1\}^t$ is the vertex set of a t -dimensional geometric hypercube in \mathbb{R}^t ; we substitute the familiar sign components $+$ and $-$ of topes by the real numbers 1 and -1 respectively. This oriented matroid is realizable as the *arrangement of coordinate hyperplanes* in \mathbb{R}^t , see [3, Example 4.1.4].

The tope graph $\mathcal{T}(\mathcal{L}(\mathcal{H}))$ of the oriented matroid \mathcal{H} is the *hypercube graph* with 2^t vertices, that is the *Hamming graph* $\mathbf{H}(t, 2)$ closely related to the *Hamming association scheme* (also called the *t-cube*) $\mathbf{H}(t, 2)$. See e.g. [1, 2, 4, 6, 7, 8, 10, 11, 12, 15, 16] on such graphs and association schemes.

The vertex set of the hypercube graph $\mathbf{H}(t, 2)$ is the tope set of \mathcal{H} , that is the collection of all words (we regard them as row vectors from \mathbb{R}^t)

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$T := (T(1), \dots, T(t)) \in \{1, -1\}^t$; vertices T' and T'' are adjacent in $\mathbf{H}(t, 2)$ iff there is a unique component $e \in E_t$ such that $T'(e) = -T''(e)$.

Recall that the vertices of $\mathbf{H}(t, 2)$ are in one-to-one correspondence with the elements of the Boolean lattice of subsets of the set E_t : it is convenient to regard the power set 2^{E_t} of the set E_t as the collection $\{T^- : T \in \{1, -1\}^t\}$ of the *negative parts* $T^- := \{e \in E_t : T(e) = -1\}$ of topes of the oriented matroid \mathcal{H} .

Let $\mathbf{R} := (R^0, R^1, \dots, R^{2t-1}, R^0)$ be a symmetric $2t$ -cycle (*symmetric cycle*, for short) in $\mathbf{H}(t, 2)$, that is, $R^{k+t} = -R^k$ for each k , $0 \leq k \leq t-1$. The vertex sequence of any $(t-1)$ -path in \mathbf{R} is a *basis* of \mathbb{R}^t ; indeed, the absolute value of the determinant of the matrix composed of those row vectors is 2^{t-1} . This in particular means that the vertex sequence $\mathfrak{V}(\mathbf{R}) := (R^0, R^1, \dots, R^{2t-1})$ of the cycle \mathbf{R} is a *maximal positive basis* of \mathbb{R}^t , see [13].

For any vertex T of $\mathbf{H}(t, 2)$ there exists a unique inclusion-minimal subset $\mathbf{Q}(T, \mathbf{R}) \subset \mathfrak{V}(\mathbf{R})$ such that

$$T = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q.$$

For all topes $T \in \{1, -1\}^t$, the linearly independent sets $\mathbf{Q}(T, \mathbf{R}) \subset \mathbb{R}^t$ are of odd cardinality; they can be explicitly described as follows [13, Cor. 2.2]:

$$\mathbf{Q}(T, \mathbf{R}) := {}_{-(T^-)}(\mathbf{max}^+({}_{-(T^-)}\mathfrak{V}(\mathbf{R}))) \quad , \quad T \in \{1, -1\}^t,$$

where ${}_{-(T^-)}\mathfrak{V}(\mathbf{R})$ is the sequence of topes obtained from the vertex sequence $\mathfrak{V}(\mathbf{R})$ by *reorientation* on the negative part T^- of T ; $\mathbf{max}^+(\cdot)$ is the subset of all topes from the resulting sequence that have *inclusion-maximal positive parts*, and the outermost operation ${}_{-(T^-)}(\cdot)$ means the reverse *reorientation* on the ground subset T^- .

If we consider the vertex set $\{1, -1\}^t$ of the hypercube graph $\mathbf{H}(t, 2)$ as the disjoint union $\dot{\bigcup}_{T \in \{1, -1\}^t} \{T\}$ of the singleton sets of its vertices, then

$$\{1, -1\}^t = \dot{\bigcup}_{T \in \{1, -1\}^t} \left\{ \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q \right\}.$$

In other words, for any symmetric cycle \mathbf{R} in $\mathbf{H}(t, 2)$ we have

$$\{1, -1\}^t = \dot{\bigcup}_{S \subseteq E_t} \left\{ \sum_{Q \in {}_{-S}(\mathbf{max}^+({}_{-S}\mathfrak{V}(\mathbf{R})))} Q \right\};$$

note that if a pair $\{P', P''\}$ is the neighborhood of a word P in the cycle ${}_{-S}\mathbf{R}$ then $P \in \mathbf{max}^+({}_{-S}\mathfrak{V}(\mathbf{R}))$ iff $|(P')^-| - 1 = |P^-| = |(P'')^-| - 1$.

In Section 2 we find the cardinalities $|\mathbf{Q}(T, \mathbf{R})|$ of the decompositions of topes, in the context of arbitrary simple oriented matroids. In Section 3 we list some metric relations for the sets $\mathbf{Q}(T, \mathbf{R})$. In Section 4 we describe a basic statistic associated with the vertices of hypercube graphs and their symmetric cycles.

2. DECOMPOSITIONS OF TOPEs WITH RESPECT TO SYMMETRIC CYCLES IN THE TOPE GRAPHS

Let $\mathcal{M} := (E_t, \mathcal{T})$ be a simple oriented matroid (it contains no loops, parallel or *antiparallel* elements) with set of tope \mathcal{T} . Let us fix in the *tope graph* of \mathcal{M} a symmetric cycle \mathbf{R} with vertex sequence $\mathfrak{V}(\mathbf{R}) := (R^0, R^1, \dots, R^{2t-1})$. If $T \in \mathcal{T}$, then we define the row *distance vector* $\mathbf{z}_{T, \mathbf{R}} := (z_{T, \mathbf{R}}(0), z_{T, \mathbf{R}}(1), \dots, z_{T, \mathbf{R}}(2t-1)) \in \ell^2(\mathbb{Z}_{2t})$ of the cycle \mathbf{R} with respect to the tope T as follows:

$$z_{T, \mathbf{R}}(k) := d(T, R^k), \quad 0 \leq k \leq 2t-1,$$

where $d(T', T'')$ denotes the *graph distance* between tope T' and T'' , that is the *Hamming distance* between the words T' and T'' . If we let $\langle T', T'' \rangle := \sum_{e \in E_t} T'(e)T''(e)$ denote the standard scalar product on \mathbb{R}^t , then $\langle T', T'' \rangle = t - 2d(T', T'')$ and, as a consequence, $d(T', T'') = \frac{1}{2}(t - \langle T', T'' \rangle)$.

Let \mathbf{I} and \mathbf{C} denote the $2t \times 2t$ *identity matrix* and *basic circulant permutation matrix*, respectively, with the rows and columns indexed from 0 to $2t-1$. If $\mathbf{v} \in \ell^2(\mathbb{Z}_{2t})$, then we denote by $\hat{\mathbf{v}} := (\hat{v}(0), \hat{v}(1), \dots, \hat{v}(2t-1))$ the *discrete Fourier transform* (see e.g. [9, 17]) of the vector \mathbf{v} . If $\mathbf{w} \in \ell^2(\mathbb{Z}_{2t})$, then we let $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle := \sum_{k=0}^{2t-1} v(k)\overline{w(k)}$ denote the complex inner product on $\ell^2(\mathbb{Z}_{2t})$, where $\overline{}$ means complex conjugation.

For any tope $T \in \mathcal{T}$, we have

$$\begin{aligned} |Q(T, \mathbf{R})| &= t - \frac{1}{8} \mathbf{z}_{T, \mathbf{R}} \cdot (2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \mathbf{z}_{T, \mathbf{R}}^\top \\ &= t - \frac{1}{4t} \sum_{k=0}^{2t-1} |\hat{z}_{T, \mathbf{R}}(k)|^2 \cdot \sin^2 \frac{\pi k}{t}, \end{aligned} \quad (2.1)$$

$$\begin{aligned} |Q(T, \mathbf{R})| &= \frac{1}{8} \mathbf{z}_{T, \mathbf{R}} \cdot (6\mathbf{I} - 4\mathbf{C}^{-1} - 4\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \mathbf{z}_{T, \mathbf{R}}^\top \\ &= \frac{1}{4t} \sum_{k=0}^{2t-1} |\hat{z}_{T, \mathbf{R}}(k)|^2 \cdot (\cos^2 \frac{\pi k}{t} - 2 \cos \frac{\pi k}{t} + 1), \end{aligned} \quad (2.2)$$

$$\begin{aligned} |Q(T, \mathbf{R})| &= \frac{t}{2} + \frac{1}{8} \mathbf{z}_{T, \mathbf{R}} \cdot (2\mathbf{I} - 2\mathbf{C}^{-1} - 2\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2) \cdot \mathbf{z}_{T, \mathbf{R}}^\top \\ &= \frac{t}{2} + \frac{1}{4t} \sum_{k=0}^{2t-1} |\hat{z}_{T, \mathbf{R}}(k)|^2 \cdot (\cos^2 \frac{\pi k}{t} - \cos \frac{\pi k}{t}) \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} |Q(T, \mathbf{R})| &= \frac{3t}{4} - \frac{1}{8} \mathbf{z}_{T, \mathbf{R}} \cdot (\mathbf{C}^{-1} + \mathbf{C} - \mathbf{C}^{-2} - \mathbf{C}^2) \cdot \mathbf{z}_{T, \mathbf{R}}^\top \\ &= \frac{3t}{4} + \frac{1}{8t} \sum_{k=0}^{2t-1} |\hat{z}_{T, \mathbf{R}}(k)|^2 \cdot (2 \cos^2 \frac{\pi k}{t} - \cos \frac{\pi k}{t} - 1), \end{aligned} \quad (2.4)$$

see [14].

Let $\alpha_{T,\mathbf{R}} \in \ell^2(\mathbb{Z}_{2t})$ denote the *autocorrelation* (see e.g. [5, Section 4.3]) of the distance vector $\mathbf{z}_{T,\mathbf{R}}$, defined by

$$\alpha_{T,\mathbf{R}}(m) := \sum_{n=0}^{2t-1} z_{T,\mathbf{R}}(n) z_{T,\mathbf{R}}((n+m) \bmod 2t), \quad 0 \leq m \leq 2t-1;$$

note that $\alpha_{T,\mathbf{R}}(k) = \alpha_{T,\mathbf{R}}(2t-k)$, $1 \leq k \leq 2t-1$. Recall that $\hat{\alpha}_{T,\mathbf{R}} = (|z_{T,\mathbf{R}}(0)|^2, |z_{T,\mathbf{R}}(1)|^2, \dots, |z_{T,\mathbf{R}}(2t-1)|^2)$.

- Let $\mathbf{b} := (2, 0, -1, 0, \dots, 0, -1, 0)$ be the first row of the matrix $2\mathbf{I} - \mathbf{C}^{-2} - \mathbf{C}^2$. Eq. (2.1) is equivalent to the relation $t - |\mathbf{Q}(T, \mathbf{R})| = \frac{1}{16t} \langle \hat{\alpha}_{T,\mathbf{R}}, \hat{\mathbf{b}} \rangle$, and Parseval's relation implies that

$$\begin{aligned} t - |\mathbf{Q}(T, \mathbf{R})| &= \frac{1}{8} \langle \alpha_{T,\mathbf{R}}, \mathbf{b} \rangle \\ &= \frac{1}{8} (2\alpha_{T,\mathbf{R}}(0) - \alpha_{T,\mathbf{R}}(2) - \alpha_{T,\mathbf{R}}(2t-2)) \\ &= \frac{1}{8} (2\alpha_{T,\mathbf{R}}(0) - 2\alpha_{T,\mathbf{R}}(2)). \end{aligned}$$

- Let $\mathbf{b} := (6, -4, 1, 0, \dots, 0, 1, -4)$ be the first row of the matrix $6\mathbf{I} - 4\mathbf{C}^{-1} - 4\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2$. Eq. (2.2) is equivalent to $|\mathbf{Q}(T, \mathbf{R})| = \frac{1}{16t} \langle \hat{\alpha}_{T,\mathbf{R}}, \hat{\mathbf{b}} \rangle$; Parseval's relation implies that

$$\begin{aligned} |\mathbf{Q}(T, \mathbf{R})| &= \frac{1}{8} \langle \alpha_{T,\mathbf{R}}, \mathbf{b} \rangle \\ &= \frac{1}{8} (6\alpha_{T,\mathbf{R}}(0) - 4\alpha_{T,\mathbf{R}}(1) + \alpha_{T,\mathbf{R}}(2) + \alpha_{T,\mathbf{R}}(2t-2) - 4\alpha_{T,\mathbf{R}}(2t-1)) \\ &= \frac{1}{8} (6\alpha_{T,\mathbf{R}}(0) - 8\alpha_{T,\mathbf{R}}(1) + 2\alpha_{T,\mathbf{R}}(2)). \end{aligned}$$

- Let $\mathbf{b} := (2, -2, 1, 0, \dots, 0, 1, -2)$ be the first row of the matrix $2\mathbf{I} - 2\mathbf{C}^{-1} - 2\mathbf{C} + \mathbf{C}^{-2} + \mathbf{C}^2$. Eq. (2.3) is equivalent to $|\mathbf{Q}(T, \mathbf{R})| - \frac{t}{2} = \frac{1}{16t} \langle \hat{\alpha}_{T,\mathbf{R}}, \hat{\mathbf{b}} \rangle$, and we have

$$\begin{aligned} |\mathbf{Q}(T, \mathbf{R})| - \frac{t}{2} &= \frac{1}{8} \langle \alpha_{T,\mathbf{R}}, \mathbf{b} \rangle \\ &= \frac{1}{8} (2\alpha_{T,\mathbf{R}}(0) - 2\alpha_{T,\mathbf{R}}(1) + \alpha_{T,\mathbf{R}}(2) + \alpha_{T,\mathbf{R}}(2t-2) - 2\alpha_{T,\mathbf{R}}(2t-1)) \\ &= \frac{1}{8} (2\alpha_{T,\mathbf{R}}(0) - 4\alpha_{T,\mathbf{R}}(1) + 2\alpha_{T,\mathbf{R}}(2)). \end{aligned}$$

- Let $\mathbf{b} := (0, -1, 1, 0, \dots, 0, 1, -1)$ be the first row of the matrix $\mathbf{C}^{-1} + \mathbf{C} - \mathbf{C}^{-2} - \mathbf{C}^2$. Eq. (2.4) is equivalent to $|\mathbf{Q}(T, \mathbf{R})| - \frac{3t}{4} = \frac{1}{16t} \langle \hat{\alpha}_{T,\mathbf{R}}, \hat{\mathbf{b}} \rangle$, and we have

$$\begin{aligned} |\mathbf{Q}(T, \mathbf{R})| - \frac{3t}{4} &= \frac{1}{8} \langle \alpha_{T,\mathbf{R}}, \mathbf{b} \rangle \\ &= \frac{1}{8} (-\alpha_{T,\mathbf{R}}(1) + \alpha_{T,\mathbf{R}}(2) + \alpha_{T,\mathbf{R}}(2t-2) - \alpha_{T,\mathbf{R}}(2t-1)) \\ &= \frac{1}{8} (-2\alpha_{T,\mathbf{R}}(1) + 2\alpha_{T,\mathbf{R}}(2)). \end{aligned}$$

We come to the conclusion:

Proposition 2.1. *Let \mathcal{M} be a simple oriented matroid, and $\mathfrak{V}(\mathbf{R})$ the vertex sequence of a symmetric cycle \mathbf{R} in the tope graph of \mathcal{M} . If T is a tope of \mathcal{M} , then for the inclusion-minimal subset $\mathbf{Q}(T, \mathbf{R}) \subset \mathfrak{V}(\mathbf{R})$ such that $T = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q$, we have*

$$|\mathbf{Q}(T, \mathbf{R})| = t - \frac{1}{4}(a_{T, \mathbf{R}}(0) - a_{T, \mathbf{R}}(2)) , \quad (2.5)$$

$$|\mathbf{Q}(T, \mathbf{R})| = \frac{1}{4}(3a_{T, \mathbf{R}}(0) - 4a_{T, \mathbf{R}}(1) + a_{T, \mathbf{R}}(2)) , \quad (2.6)$$

$$|\mathbf{Q}(T, \mathbf{R})| = \frac{t}{2} + \frac{1}{4}(a_{T, \mathbf{R}}(0) - 2a_{T, \mathbf{R}}(1) + a_{T, \mathbf{R}}(2)) \quad (2.7)$$

and

$$|\mathbf{Q}(T, \mathbf{R})| = \frac{3t}{4} + \frac{1}{4}(-a_{T, \mathbf{R}}(1) + a_{T, \mathbf{R}}(2)) , \quad (2.8)$$

where $a_{T, \mathbf{R}}$ is the autocorrelation of a distance vector of the cycle \mathbf{R} with respect to the tope T .

3. BASIC METRIC PROPERTIES OF DECOMPOSITIONS

In this section we consider a simple oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$ with distinguished symmetric cycle \mathbf{R} in its tope graph. If $T \in \mathcal{T}$, then we have

$$\sum_{Q \in \mathbf{Q}(T, \mathbf{R})} d(T, Q) = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} \frac{1}{2}(t - \langle T, Q \rangle) = \frac{1}{2}|\mathbf{Q}(T, \mathbf{R})|t - \frac{1}{2} \underbrace{\sum_{Q \in \mathbf{Q}(T, \mathbf{R})} \langle T, Q \rangle}_{\|T\|^2 = t} .$$

Remark 3.1. *For any tope T of \mathcal{M} we have*

$$\sum_{Q \in \mathbf{Q}(T, \mathbf{R})} d(T, Q) = \frac{1}{2}(|\mathbf{Q}(T, \mathbf{R})| - 1)t \quad (3.1)$$

and

$$|\mathbf{Q}(T, \mathbf{R})| = 1 + \frac{2}{t} \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} d(T, Q) . \quad (3.2)$$

Now suppose that $T \notin \mathfrak{V}(\mathbf{R})$ and $\mathbf{Q}(T, \mathbf{R}) := \{Q^1, \dots, Q^{|\mathbf{Q}(T, \mathbf{R})|}\}$. Note that

$$\begin{aligned} t = \|T\|^2 = \langle T, T \rangle &= \left\langle \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q, \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q \right\rangle \\ &= |\mathbf{Q}(T, \mathbf{R})|t + 2 \sum_{1 \leq i < j \leq |\mathbf{Q}(T, \mathbf{R})|} (t - 2d(Q^i, Q^j)) \\ &= |\mathbf{Q}(T, \mathbf{R})|t + 2(|\mathbf{Q}(T, \mathbf{R})|t - 4 \sum_{1 \leq i < j \leq |\mathbf{Q}(T, \mathbf{R})|} d(Q^i, Q^j)) . \end{aligned}$$

Remark 3.2. For a tope T of \mathcal{M} , such that $T \notin \mathfrak{V}(\mathbf{R})$, we have

$$\sum_{1 \leq i < j \leq |\mathbf{Q}(T, \mathbf{R})|} d(Q^i, Q^j) = \frac{1}{4}(|\mathbf{Q}(T, \mathbf{R})|^2 - 1)t, \quad (3.3)$$

$$|\mathbf{Q}(T, \mathbf{R})| = \sqrt{1 + 4 \frac{\sum_{i < j} d(Q^i, Q^j)}{t}}. \quad (3.4)$$

Moreover,

$$\sum_{1 \leq i < j \leq |\mathbf{Q}(T, \mathbf{R})|} d(Q^i, Q^j) = \frac{1}{2}(|\mathbf{Q}(T, \mathbf{R})| + 1) \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} d(T, Q). \quad (3.5)$$

4. SYMMETRIC CYCLES IN THE HYPERCUBE GRAPHS

We now turn to an investigation of the combinatorial properties of the vertex set $\{1, -1\}^t$ of the hypercube graph $\mathbf{H}(t, 2)$ and its symmetric cycles. Let \mathbf{R} be a symmetric cycle, with its distance vectors $\mathbf{z}_{T, \mathbf{R}}$, $T \in \{1, -1\}^t$.

According to Eq. (2.5), we have

$$\begin{aligned} \sum_{T \in \{1, -1\}^t} |\mathbf{Q}(T, \mathbf{R})| &= \sum_{T \in \{1, -1\}^t} \left(t - \frac{1}{4} \sum_{n=0}^{2t-1} z_{T, \mathbf{R}}(n) (z_{T, \mathbf{R}}(n) - z_{T, \mathbf{R}}((n+2) \bmod 2t)) \right) \\ &= 2^t t - \frac{1}{4} \sum_{n=0}^{2t-1} \sum_{T \in \{1, -1\}^t} z_{T, \mathbf{R}}(n) (z_{T, \mathbf{R}}(n) - z_{T, \mathbf{R}}((n+2) \bmod 2t)) \\ &= 2^t t - \frac{1}{2} t \sum_{T \in \{1, -1\}^t} z_{T, \mathbf{R}}(n) (z_{T, \mathbf{R}}(n) - z_{T, \mathbf{R}}((n+2) \bmod 2t)) \\ &= t \left(2^t - \frac{1}{2} \sum_{0 \leq i, j \leq t} \mathbf{p}_{ij}^2 i(i-j) \right), \end{aligned}$$

where for any vertices $X, Y \in \{1, -1\}^t$, such that $d(X, Y) = 2$, the quantity $|\{Z \in \{1, -1\}^t : d(Z, X) = i, d(Z, Y) = j\}| =: \mathbf{p}_{ij}^2$ is the same; this is an *intersection number* of the Hamming association scheme $\mathbf{H}(t, 2)$, see e.g. [12, §21.3]. Thus, we have

$$\begin{aligned} \sum_{T \in \{1, -1\}^t} |\mathbf{Q}(T, \mathbf{R})| &= t \left(2^t - \frac{1}{2} \sum_{\substack{1 \leq i \leq t; \\ j \in \{i-2, i+2\}; \\ 0 \leq j \leq t}} \binom{t-2}{\frac{i+j}{2}-1} \underbrace{\binom{2}{\frac{i-j}{2}+1}}_{1 \text{ when } j \in \{i-2, i+2\}} i(i-j) \right) \\ &= t \left(2^t - \frac{1}{2} \sum_{\substack{1 \leq i \leq t; \\ j \in \{i-2, i+2\}; \\ 0 \leq j \leq t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j) \right). \end{aligned}$$

Since

$$\mathfrak{s}(t) := \sum_{\substack{1 \leq i \leq t; \\ j \in \{i-2, i+2\}: 0 \leq j \leq t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j) = 2\mathfrak{s}(t-1) = 2^t ,$$

we come to the following conclusion:

Remark 4.1. *Let \mathbf{R} be a symmetric cycle in the hypercube graph $\mathbf{H}(t, 2)$.*

(i) *We have*

$$\begin{aligned} \sum_{T \in \{1, -1\}^t} |\mathbf{Q}(T, \mathbf{R})| &= \frac{t}{2} \sum_{\substack{1 \leq i \leq t; \\ j \in \{i-2, i+2\}: \\ 0 \leq j \leq t}} \mathfrak{p}_{ij}^2 i(i-j) \\ &= \frac{t}{2} \sum_{\substack{1 \leq i \leq t; \\ j \in \{i-2, i+2\}: \\ 0 \leq j \leq t}} \binom{t-2}{\frac{i+j}{2}-1} i(i-j) , \end{aligned} \quad (4.1)$$

that is,

$$\sum_{T \in \{1, -1\}^t} |\mathbf{Q}(T, \mathbf{R})| = 2^{t-1} t \quad (4.2)$$

—the number of edges in $\mathbf{H}(t, 2)$.

(ii) *Eqs. (4.2) and (3.1) yield*

$$\sum_{T \in \{1, -1\}^t} \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} d(T, Q) = 2^{t-2} (t-2)t . \quad (4.3)$$

If j is an odd integer, $1 \leq j \leq t$, then define

$$c_j(t) := |\{T \in \{1, -1\}^t : |\mathbf{Q}(T, \mathbf{R})| = j\}| ;$$

thus, we have

$$\sum_{\substack{1 \leq j \leq t: \\ j \text{ odd}}} c_j(t) = 2^t .$$

Since

$$\sum_{\substack{1 \leq j \leq t: \\ j \text{ odd}}} j c_j(t) := \sum_{T \in \{1, -1\}^t} |\mathbf{Q}(T, \mathbf{R})| = 2^{t-1} t = \sum_{0 \leq i \leq t} i \binom{t}{i} ,$$

by Remark 4.1(i), the quantities $c_j(t)$ are read off from Eq. (4.2):

Theorem 4.2. *Let \mathbf{R} be a symmetric cycle, with vertex set $\mathfrak{V}(\mathbf{R})$, in the hypercube graph $\mathbf{H}(t, 2)$. Consider, for the vertices $T \in \{1, -1\}^t$ of the graph $\mathbf{H}(t, 2)$, the inclusion-minimal subsets $\mathbf{Q}(T, \mathbf{R}) \subset \mathfrak{V}(\mathbf{R})$ such that $T = \sum_{Q \in \mathbf{Q}(T, \mathbf{R})} Q$.*

For any odd integer j , $1 \leq j \leq t$, we have

$$c_j(t) := |\{T \in \{1, -1\}^t : |\mathbf{Q}(T, \mathbf{R})| = j\}| = 2 \binom{t}{j} . \quad (4.4)$$

In other words, the polynomial

$$\gamma_t(x) := \sum_{\substack{1 \leq j \leq t: \\ j \text{ odd}}} c_j(t) x^j ,$$

in the variable x , is

$$\gamma_t(x) = 2 \sum_{\substack{1 \leq j \leq t: \\ j \text{ odd}}} \binom{t}{j} x^j . \quad (4.5)$$

We can define $\gamma_1(x)$ as $2x$. The polynomials $\gamma_t(x)$, where $2 \leq t \leq 10$, are collected in the following table:

t	$\gamma_t(x) := \sum_{T \in \{1, -1\}^t} x^{ \mathbf{Q}(T, \mathbf{R}) }$
2	$4x$
3	$6x + 2x^3$
4	$8x + 8x^3$
5	$10x + 20x^3 + 2x^5$
6	$12x + 40x^3 + 12x^5$
7	$14x + 70x^3 + 42x^5 + 2x^7$
8	$16x + 112x^3 + 112x^5 + 16x^7$
9	$18x + 168x^3 + 252x^5 + 72x^7 + 2x^9$
10	$20x + 240x^3 + 504x^5 + 240x^7 + 20x^9$

In view of the simplicity of the statistic on the vertices and symmetric cycles of the hypercube graphs, given by the polynomials $\gamma_t(x)$, there are many ways to describe the binomial-type combinatorial properties of the quantities $c_j(t)$. For instance, if t is even, then for any odd integer j , $1 \leq j < t$, we have $c_j(t) = c_{t-j}(t)$; if t is odd, then for any odd integer j , $1 \leq j \leq t$, we have $jc_j(t) = (1 + t - j)c_{1+t-j}(t)$, and so on.

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